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VIBRATIONS AND STABILITY OF A TUBE CONVEYING FLUID

by

S. S. Chen and G. S. Rosenberg

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Engineering and Technology Division

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NOMENCLATURE

a_{nj}	Coupling coefficient
$\bar{A}_1$, $\bar{A}_2$	Amplitude
c	External damping coefficient
e	Strain
E	Young's modulus
f_i	Fourier expansion coefficient of initial imperfection
F	Transverse force per unit length
g	Initial displacement
g_0	Dimensionless initial displacement
h	Initial velocity
h_0	Dimensionless initial velocity
I	Moment of inertia of the tube
k	Wave number
$\bar{k}_n$	Defined in Eqs. 26
ℓ	Length of the tube
M	Moment
M.F.	Magnification factor
m_f	Added mass per unit length
m_t	Mass per unit length of the tube
P	Period $(2\pi/\omega)$
q	External loading per unit length
q_n, $q_n^{(i)}$	Dimensionless time coordinate
Q	Dimensionless force
Q_n	Generalized dimensionless force
$\bar{r}$	$U/\bar{v}$
r_n	u_0/u_n
R	$q/(m_t + m_f)$
t	Time
T	Axial force
u	Dimensionless fluid transport velocity
u_0	Steady component of u
u_n	Critical fluid transport velocity
U	Fluid transport velocity
v	Phase velocity
$\bar{v}$	Wave velocity for U = 0
$\bar{v}_1$, $\bar{v}_2$	Phase velocities given in Eqs. 16
$\bar{v}_p$	Phase-propagating velocity
w	Dimensionless transverse displacement of the tube
w_0	Dimensionless initial deflection = y_0/ℓ
W	Work
x	Axial coordinate
y	Transverse displacement of the tube
y_0	Initial deflection
z	Distance from each fiber to neutral axis

Greek Letters

α, δ	Dimensionless damping coefficient
β	$[m_f/(m_t + m_f)]^{1/2}$
Γ	Dimensionless axial force
δ_{nj}	Kronecker Delta
ϵ	$2\beta u$ ($2\beta u_0$, if u is not a constant)
θ_n	Arbitrary phase constant
ζ, ζ_n	Damping coefficient
λ	Coefficient of internal damping
$\bar{\lambda}_1$, $\bar{\lambda}_2$	Wave number defined by Eqs. 22
μ	Excitation parameter
ν_n, $\bar{\nu}_n$	Natural frequency for neglecting Coriolis force and u = 0
ξ	x/ℓ
ρ_1, ρ_2, ρ_3	Parameters defined by Eqs. 60
σ	Stress
τ	Dimensionless time
ϕ_n	Orthonormal eigenfunction for neglecting the Coriolis force term
ω	Dimensionless frequency
ω_n, $\bar{\omega}_n$	Natural frequency
Ω_n	Natural frequency for neglecting the Coriolis force

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ABSTRACT

This report analyzes the free vibration, forced vibration, and parametric response of a simply supported tube conveying fluid; the effects of initial curvature and Coriolis force are included and evaluated. The equation of motion is formulated to incorporate initial curvature, damping, axial force, and unsteady flow. Galerkin's method is used to determine the free vibration by solving an algebraic characteristic equation; the forced vibration is reduced to a system of coupled ordinary differential equations. Excitation due to pulsating flow is also examined by means of Galerkin's method, which, in this case, yields a set of coupled Mathieu-Hill-type equations. The possibility of onset of a parametric behavior is demonstrated, and explicit formulas are obtained for the boundaries of the first two instability regions. Due to the presence of a Coriolis force, the system is characterized by a phase difference and mode coupling. The significance of the Coriolis force is discussed. The effect of initial curvature is analyzed, and wave propagation and vibrational problems in the media without flexural rigidity are also investigated in detail.

I. INTRODUCTION

Various components of reactors, such as fuel pins, control rods, and heat-exchanger tubes, are long, slender members having some potential for vibration. Structural vibrations often cause damage to such components through wear and fatigue.

One source of energy that can induce these vibrations is the high-velocity fluid flowing through a reactor core. Thus the vibration of tubes exposed to a parallel flow is of practical importance in reactor applications.

The transverse vibration of a tube conveying fluid has received considerable attention since the early 1950's. Ashley and Haviland[1] were among the first investigators. Housner[2] showed the existence of a critical flow velocity and revised several of the conclusions given previously.[1]

The relationship between natural frequencies and fluid transport velocity was analyzed by Niordson,[3] who derived the equation of motion from shell theory for a long-wave approximation. Long[4] calculated the frequency by a power-series method and performed the experimental investigation. Later, Benjamin[5] analyzed the stability of a chain of articulated pipes; the stability of a cantilever pipe was studied theoretically and experimentally by Gregory and Paidoussis.[6] More recently, an exact solution for the natural frequencies and axial distribution of phase was computed by Naguleswaran and Williams.[7] And most recently, Thurman and Mote[8] presented a nonlinear oscillation study to assess the applicable range of linear theory.

The present study is within the framework of linear theory. However, an analytical solution is difficult to obtain because the governing differential equation of motion is not self-adjoint, due to the existence of the Coriolis force, which creates a mixed derivative $\partial^2 y/\partial x \partial t$ in the equation (see Eq. 5). Accordingly, the first objective is to investigate the significance and effects of the Coriolis force on free vibration, forced motion, and parametric response.

The influence of unsteady flow has not been analyzed; moreover, the tubes treated by the referenced investigators were assumed to be initially straight. However, the fluid transport velocity is not necessarily steady, and the tubes may have some initial curvature. Therefore, the second objective is to study the effects of pulsating flow and initial curvature.

The third objective is to study the overall dynamic behavior and its physical origins and effects, such that one may have a better picture of the system characteristics.

First, the equation of motion is formulated to include the effects of viscous damping, axial force, initial curvature, and unsteady flow. As a special case, the system without flexural rigidity is first studied in detail. Then, free vibration and forced vibration are analyzed, using Galerkin's method and the mode-shape functions obtained for zero Coriolis force. The resulting algebraic characteristic equation is easily solved and gives accurate results; forced vibration is reduced to a system of coupled differential equations. The parametric response is also examined by Galerkin's method, which, in this case, yields a system of Mathieu-Hill equations. The boundary equations of instability are obtained for the first two regions. Finally, the significance of the Coriolis force is discussed in terms of energy and transient response. The effects of initial curvature are investigated by neglecting the Coriolis force.

II. STATEMENT OF THE PROBLEMS

A. Governing Equation

The system under consideration is a tube conveying fluid whose velocity is U (see Fig. 1). The tube is of uniform cross section and has a mass per unit length m_t. We shall assume that the tube has initial deflection $y_0(x)$ and initial axial tension T, and that the material obeys a stress-strain relationship of the Kelvin type; i.e.,

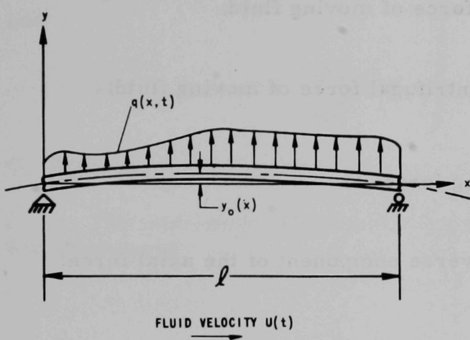

Fig. 1. Simply Supported Tube Conveying Fluid. ANL Neg. No. 113-3637.

$$\sigma = Ee + \lambda\dot{e}, \tag{1}$$

where E is Young's modulus and λ is the internal damping coefficient. It is known[3] that if the wavelength is large in comparison with the diameter of the tube, the Bernoulli-Euler beam theory can be applied to study the vibration of the tube; under such a condition, the tube satisfies the assumption of plane sections remaining plane. From the classical bending theory, we have

$$\left.\begin{aligned} \sigma &= \frac{Mz}{I} \\ \text{and} \qquad & \\ e &= -z\frac{\partial^2 y}{\partial x^2}. \end{aligned}\right\} \tag{2}$$

The equation of motion may be written

$$\frac{\partial^2 M}{\partial x^2} = F, \tag{3}$$

where F is the resultant lateral force exerted on the tube and is given by

$$F = (m_t + m_f)\frac{\partial^2 y}{\partial t^2} + 2m_f U\frac{\partial^2 y}{\partial x \partial t} + m_f U^2\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y_0}{\partial x^2}\right)$$

$$+ c\frac{\partial y}{\partial t} - T\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y_0}{\partial x^2}\right) - q. \tag{4}$$

The physical interpretation of each term in Eq. 4 is as follows:

$(m_t + m_f) \dfrac{\partial^2 y}{\partial t^2}$ is the inertia force of tube and fluid;

$2m_t U \dfrac{\partial^2 y}{\partial x \partial t}$ is the Coriolis force of moving fluid;

$m_f U^2 \left(\dfrac{\partial^2 y}{\partial x^2} + \dfrac{\partial^2 y_0}{\partial x^2} \right)$ is the centrifugal force of moving fluid;

$c \dfrac{\partial y}{\partial t}$ is the external damping force;

$T \left(\dfrac{\partial^2 y}{\partial x^2} + \dfrac{\partial^2 y_0}{\partial x^2} \right)$ is the transverse component of the axial force;

and

q is the external force applied on the tube.

On substituting Eqs. 1, 2, and 4 into Eq. 3, we obtain the equation of motion:

$$EI \frac{\partial^4 y}{\partial x^4} + \lambda I \frac{\partial^5 y}{\partial x^4 \partial t} + m_f U^2 \frac{\partial^2 y}{\partial x^2} + 2m_f U \frac{\partial^2 y}{\partial x \partial t}$$

$$- T \frac{\partial^2 y}{\partial x^2} + (m_t + m_f) \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t}$$

$$= q - (m_f U^2 - T) \frac{\partial^2 y_0}{\partial x^2}. \qquad (5)$$

B. Initial and Boundary Conditions

The initial conditions of the system are specified as

$$\left. \begin{aligned} y &= g(x) \quad \text{at } t = 0; \\ &\text{and} \\ \frac{\partial y}{\partial t} &= h(x) \quad \text{at } t = 0. \end{aligned} \right\} \qquad (6)$$

For the simply supported tube discussed in this study, the proper boundary conditions are:

$$\left.\begin{aligned} &\text{at } x = 0:\; y = 0,\; \frac{\partial^2 y}{\partial x^2} = 0; \\ &\text{and} \\ &\text{at } x = \ell:\; y = 0,\; \frac{\partial^2 y}{\partial x^2} = 0. \end{aligned}\right\} \qquad (7)$$

C. Dimensionless Parameters

For analytical convenience, the following diminsionless quantities are introduced:

$$\left.\begin{aligned} \xi &= x/\ell, \\ w &= y/\ell, \\ w_0 &= y_0/\ell, \\ g_0 &= g/\ell, \\ h_0 &= h\ell[(m_t + m_f)/EI]^{1/2}, \\ \alpha &= \left[\frac{I}{E(m_t + m_f)}\right]^{1/2} \frac{\lambda}{\ell^2}, \\ \beta &= \left(\frac{m_f}{m_t + m_f}\right)^{1/2}, \\ \delta &= \frac{c\ell^2}{[EI(m_t + m_f)]^{1/2}}, \\ Q &= \frac{q\ell^3}{EI}, \\ \Gamma &= T\ell^2/EI, \\ u &= \left(\frac{m_f}{EI}\right)^{1/2} U\ell, \\ &\text{and} \\ \tau &= \left(\frac{EI}{m_t + m_f}\right)^{1/2} \frac{t}{\ell^2}. \end{aligned}\right\} \qquad (8)$$

On substituting from Eqs. 8 into Eqs. 5-7, we obtain the equation of motion with corresponding initial and boundary conditions:

$$\frac{\partial^4 w}{\partial \xi^4} + \alpha \frac{\partial^5 w}{\partial \xi^4 \partial \tau} + (u^2 - \Gamma) \frac{\partial^2 w}{\partial \xi^2} + 2\beta u \frac{\partial^2 w}{\partial \xi \partial \tau} + \delta \frac{\partial w}{\partial \tau} + \frac{\partial^2 w}{\partial \tau^2} =$$

$$Q - (u^2 - \Gamma) \frac{\partial^2 w_0}{\partial \xi^2}. \tag{9}$$

The initial conditions are:

$$\left.\begin{aligned} & w = g_0 \quad \text{at } \tau = 0; \\ \text{and} & \\ & \frac{\partial w}{\partial \tau} = h_0 \quad \text{at } \tau = 0. \end{aligned}\right\} \tag{10}$$

The boundary conditions are reduced to

$$\left.\begin{aligned} & w = \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{at } \xi = 0; \\ \text{and} & \\ & w = \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{at } \xi = 1. \end{aligned}\right\} \tag{11}$$

Except in the parametric response and initial curvature studies, the fluid transport velocity u is taken as constant; and except for the initial curvature and forced-vibration studies, the tube is assumed to be perfectly straight. Equations 5-7 or 9-11 are the complete mathematical statements of the system. Also, dependent on the particular problem of concern, those terms that are anticipated to be insignificant will not be included in the analysis.

III. EXACT SOLUTION FOR THE SYSTEM WITH ZERO RIGIDITY

A. Unbounded System

If the flexural rigidity is neglected, Eq. 5 reduces to an equation of the second order; further assuming no damping, we obtain

$$(\beta^2 U^2 - \bar{v}^2) \frac{\partial^2 y}{\partial x^2} + 2\beta^2 U \frac{\partial^2 y}{\partial x \partial t} + \frac{\partial^2 y}{\partial t^2} = R(x,t), \tag{12}$$

where

$$\left.\begin{aligned} \beta^2 &= \frac{m_f}{m_f + m_t}, \\ \bar{v} &= \left(\frac{T}{m_t + m_f}\right)^{1/2}, \\ \text{and} \\ R &= \frac{q}{m_t + m_f}. \end{aligned}\right\} \tag{13}$$

Note that $\bar{v}$ is the phase velocity of a wave when U is equal to zero. When $\beta = 1$, Eq. 12 is the equation of motion for a moving string, bandsaw, threadline, or transmission chain and thus can be used to shed some light on the dynamic behavior of such moving systems in transverse vibration. For this reason, it is of practical interest and value to study this special case.

From the discriminant of Eq. 12, it may be seen that if

$$U < \frac{\bar{v}}{\sqrt{\beta^2(1-\beta^2)}}, \tag{14}$$

the equation is of the hyperbolic type; thus it is possible to seek either a progressive-wave solution or a "standing-wave" solution. There are two different wave velocities. The general solution of Eq. 12 for $R = 0$ is

$$y = G(x - \bar{v}_1 t) + H(x + \bar{v}_2 t), \tag{15}$$

where

$$\left.\begin{aligned} \bar{v}_1 &= [\bar{v}^2 - \beta^2 U^2(1-\beta^2)]^{1/2} + \beta^2 U, \\ \bar{v}_2 &= [\bar{v}^2 - \beta^2 U^2(1-\beta^2)]^{1/2} - \beta^2 U, \end{aligned}\right\} \tag{16}$$

and G and H are arbitrary functions. G corresponds to a plane wave which is always propagating downstream (in the same direction as U); and H corresponds to a plane wave which may travel upstream or downstream, depending on the transport velocity U. In the subcritical case ($\bar{v} > \beta U$), both waves propagate in different directions and with different velocities. In the supercritical case ($\bar{v} < \beta U$), both waves propagate in the same direction, but with different velocities. And in the critical case ($\bar{v} = \beta U$), there is only one propagating wave whose velocity is $2\bar{v}$; the other is a standing wave. In all cases, $\bar{v}_1$ is always faster than $\bar{v}_2$; this reflects a tendency for the transport velocity U to accelerate the wave G, but to retard the wave H.

The disturbance in these systems is propagated in a different manner from that in a stationary string. Here, the initial value problem of the Cauchy type is considered. The system whose motion is governed by Eq. 12 is subjected to an initial disturbance specified by Eqs. 6; we are interested in the response. The system is nondispersive, and its solution can be obtained by D'Alembert's method. The result is given as follows (see Appendix A):

For $\bar{v} \neq \beta U$,

$$y(x,t) = \frac{1}{\bar{v}_1 + \bar{v}_2}\left[\bar{v}_1 g(x+\bar{v}_2 t) + \bar{v}_2 g(x-\bar{v}_1 t) + \int_{x-\bar{v}_1 t}^{x+\bar{v}_2 t} h(s)\,ds + \int_0^t d\tau \int_{x-\bar{v}_1(t-\tau)}^{x+\bar{v}_2(t-\tau)} R(s,\tau)\,ds\right]. \tag{17a}$$

For $\bar{v} = \beta U$,

$$y(x,t) = g(x) + \frac{1}{2\bar{v}}\left[\int_x^{x-2\bar{v}t} h(s)\,ds + \int_0^t d\tau \int_{x-2\bar{v}(t-\tau)}^{x} R(s,\tau)\,ds\right]. \tag{17b}$$

To illustrate the effect of transport velocity on wave propagation, two examples are given.

Initial Disturbance (R = 0)

The initial data are taken as

$$\left.\begin{aligned} g &= \exp(-x^2) \\ &\text{and} \\ h &= 0. \end{aligned}\right\} \tag{18}$$

Figure 2 shows the propagation of an initial disturbance, using these values; the effect of U on the propagation is easily seen. At U = 0, the waves propagate at the same speed. But for $U \neq 0$, the waves have different magnitudes of phase velocity and wave form: The wave with larger phase velocity has smaller amplitude. In the supercritical case (Fig. 2c), both waves propagate downstream and one has negative magnitude.

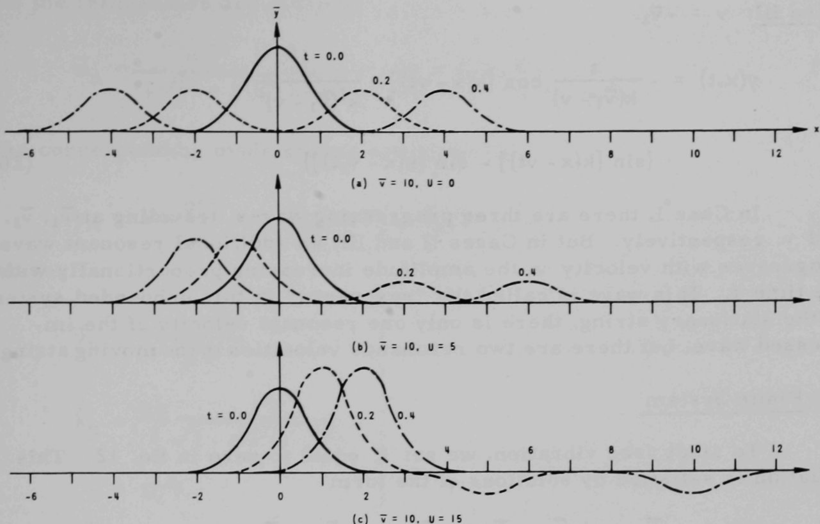

Fig. 2. Propagation of Initial Disturbance in an Unbounded System (β = 1.0). ANL Neg. No. 113-3636.

Impressed Wave of Force (g = 0, h = 0)

Consider the case in which a train of sinusoidal waves of force per unit length acting on the system is given by

$$R(x,t) = \sin [k(x - vt)]. \tag{19}$$

The wave represented by this equation travels in the x-direction with wave number k and circular frequency kv. The general solution to this problem is obtained by substituting Eq. 19 into Eqs. 17. Three cases must be distinguished.

Case I: $v \neq \bar{v}_1$, $v \neq \bar{v}_2$.

$$y(x,t) = \frac{1}{k^2(\bar{v}_1 + \bar{v}_2)} \left\{ \frac{\bar{v}_1 + \bar{v}_2}{(\bar{v}_2 + v)(\bar{v}_1 - v)} \sin [k(x - vt)] \right.$$

$$\left. - \frac{1}{\bar{v}_2 + v} \sin [k(x + \bar{v}_2 t)] - \frac{1}{\bar{v}_1 - v} \sin [k(x - \bar{v}_1 t)] \right\} \tag{20a}$$

Case II: $v = \bar{v}_1$.

$$y(x,t) = \frac{t}{k(\bar{v}_2 + v)} \cos [k(x - vt)] + \frac{1}{k^2(\bar{v}_2 + v)^2}$$

$$\cdot \{\sin [k(x - vt)] - \sin [k(x + \bar{v}_2 t)]\} \tag{20b}$$

Case III: $v = -\bar{v}_2$.

$$y(x,t) = -\frac{t}{k(\bar{v}_1 - v)} \cos [k(x - vt)] + \frac{1}{k^2(\bar{v}_1 - v)^2}$$

$$\cdot \{\sin [k(x - vt)] - \sin [k(x - v_1 t)]\} \tag{20c}$$

In Case I, there are three progressing waves, traveling at $\bar{v}_1$, $\bar{v}_2$, and v, respectively. But in Cases II and III, an additional resonant wave progresses with velocity v, the amplitude increasing proportionally with the time t. This wave is called the "resonance" in this unbounded system. In the stationary string, there is only one resonant velocity of the impressed wave, but there are two resonance velocities in the moving string.

B. Finite System

To study free vibration, we set R equal to zero in Eq. 12. This equation is satisfied by solutions of the form

$$y(x,t) = \bar{A}_1 \exp[i(\bar{\lambda}_1 x + \bar{\omega} t)] + \bar{A}_2 \exp[i(\bar{\lambda}_2 x + \bar{\omega} t)], \tag{21}$$

provided that

$$\left.\begin{aligned} \bar{\lambda}_1 &= \frac{\bar{\omega}}{\bar{v}^2 - \beta^2 U^2} \left[\beta^2 U + \sqrt{\bar{v}^2 - \beta^2 U^2(1 - \beta^2)}\right], \\ &\text{and} \\ \bar{\lambda}_2 &= \frac{\bar{\omega}}{\bar{v}^2 - \beta^2 U^2} \left[\beta^2 U - \sqrt{\bar{v}^2 - \beta^2 U^2(1 - \beta^2)}\right]. \end{aligned}\right\} \tag{22}$$

From knowledge of the boundary conditions,

$$\left.\begin{aligned} y(x,t) &= 0 \quad \text{at } x = 0, \\ &\text{and} \\ y(x,t) &= 0 \quad \text{at } x = \ell, \end{aligned}\right\} \tag{23}$$

the values of $\bar{\lambda}_1$ and $\bar{\lambda}_2$ corresponding to the natural frequencies of system vibration, and the values taken by $\bar{A}_1$ and $\bar{A}_2$, may be determined. The frequency equation is obtained as

$$\sin (\bar{\lambda}_1 - \bar{\lambda}_2) = 0,$$

and the frequencies are given by

$$\overline{\omega}_n = \frac{n(\overline{v}^2 - \beta^2 U^2)}{\ell\sqrt{\overline{v}^2 - \beta^2 U^2(1-\beta^2)}}, \qquad n = 1, 2, 3 \ldots . \tag{24}$$

The corresponding mode shapes are given by

$$y_n = \sin\frac{n\pi x}{\ell} \cos\left[\overline{k}_n(x + \overline{v}_p t + \overline{\theta}_n)\right], \tag{25}$$

where

$$\left.\begin{aligned} \overline{v}_p &= \overline{v}\,\frac{1 - \beta^2\overline{r}^2}{\beta^2\overline{r}}, \\ \overline{k}_n &= \frac{n\pi}{\ell}\,\frac{\beta^2\overline{r}}{\sqrt{1 - \beta^2\overline{r}^2(1-\beta^2)}}, \\ \overline{r} &= U/\overline{v}, \end{aligned}\right\} \tag{26}$$

and $\overline{\theta}_n$ is an arbitrary constant.

The natural frequencies depend on the transport velocity, U; their relationships are given in Fig. 3. Here, ω^* is defined by

$$\omega^* = \frac{\overline{\omega}_n}{\overline{\nu}_n}, \tag{27}$$

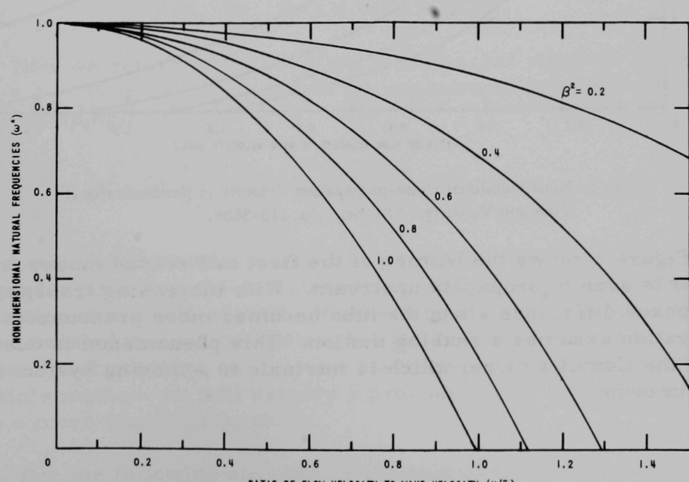

Fig. 3. Nondimensional Natural Frequencies vs Nondimensional Transport Velocity. ANL Neg. No. 113-3632.

where $\bar{\nu}_n$ is the natural frequency for zero transport velocity. The ratio is independent of n, because the media are nondispersive. The distinguishing feature of the system is that the phase is not constant along the axis for each mode, but propagates with a velocity $\bar{v}_p$.

The phase-propagating velocity is a function of transport velocity, as shown in Fig. 4. For $U = 0$, $\bar{v}_p$ is infinite and the system vibrates in the same phase. But when $U \neq 0$, the system does not possess the classical normal modes, because various parts of the system do not pass through their equilibrium configuration at the same instant of time.

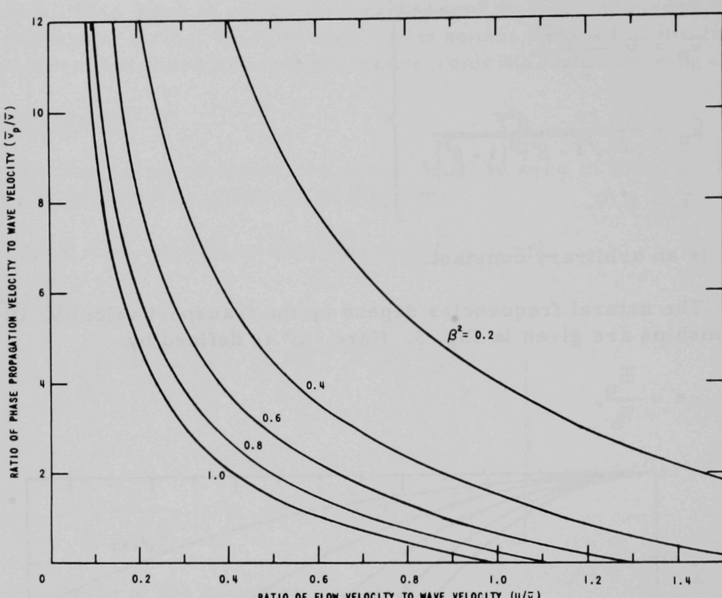

Fig. 4. Nondimensional Phase-propagation Velocity vs Nondimensional Transport Velocity. ANL Neg. No. 113-3638.

Figure 5 shows the history of the first and second modes from which the phase is seen to propagate upstream. With increasing transport velocity, the phase difference along the tube becomes more pronounced, and the tube vibration assumes a snaking motion. This phenomenon is due to the effect of the Coriolis force, which is intrinsic to a moving system in transverse vibration.

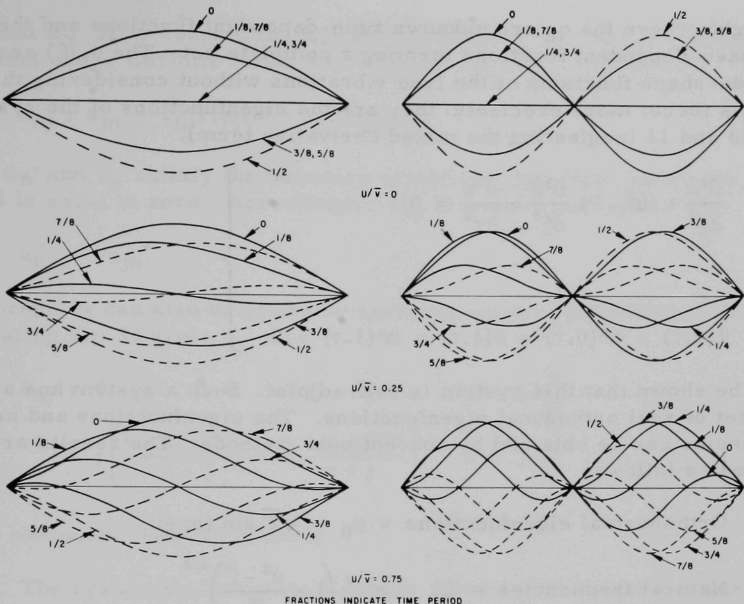

Fig. 5. Variation of Amplitudes of Fundamental Mode and Second Mode during a Period. ANL Neg. No. 113-3633.

IV. FREE VIBRATION

Now we return to the problem with flexural rigidity. By setting $Q = \alpha = \delta = 0$, in Eq. 9, we obtain the equation of motion for free vibration (no damping); i.e.,

$$\frac{\partial^4 w}{\partial \xi^4} + (u^2 - \Gamma)\frac{\partial^2 w}{\partial \xi^2} + 2\beta u \frac{\partial^2 w}{\partial \xi \partial \tau} + \frac{\partial^2 w}{\partial \tau^2} = 0, \qquad (28)$$

with its associated boundary conditions given by Eqs. 11. The difficulty in this boundary-value problem arises from the mixed derivative term; i.e., the equation is not self-adjoint. Niordson[3] and Naguleswaran and Williams[7] have formulated the exact solution, but the characteristic equation is transcendental and requires an iterative technique for solution. By using Galerkin's method, we will develop a procedure that transforms the problem to a more tractable form.

For the following analysis, a solution of the form

$$w(\xi,\tau) = \sum_n q_n(\tau)\phi_n(\xi) \qquad (29)$$

is sought, where the q_n are unknown time-dependent functions and the $\phi_n(\xi)$ are space-dependent functions forming a complete set. The $\phi_n(\xi)$ are chosen as mode-shape functions of the free vibrations without considering the Coriolis force; more precisely, they are the eigenfunctions of the system of Eqs. 28 and 11 (neglecting the mixed derivative term),

$$\left.\begin{aligned} &\frac{\partial^4\phi}{\partial\xi^4} + (u^2 - \Gamma)\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\tau^2} = 0 \\ &\text{and} \\ &\phi(0,\tau) = \phi''(0,\tau) = \phi(1,\tau) = \phi''(1,\tau) = 0. \end{aligned}\right\} \tag{30}$$

It can be shown that this system is self-adjoint. Such a system has a complete set of real orthogonal eigenfunctions. The eigenfunctions and natural frequencies can be obtained by conventional methods. The results are summarized as follows:

$$\text{Orthonormal eigenfunctions} = \phi_n = \sqrt{2}\,\sin(n\pi\xi); \tag{31a}$$

$$\text{Natural frequencies} = \Omega_n = n^2\pi^2\left(1 - \frac{u^2 - \Gamma}{u_n^2}\right)^{1/2}; \tag{31b}$$

and

$$\text{Critical fluid transport velocity} = u_n = n\pi; \tag{31c}$$

where u_n is the critical flow velocity when the tube buckles or the frequency becomes zero.

Having an orthonormal set of functions, the Galerkin orthogonality condition is used to obtain the function q_n. On substituting Eq. 29 into Eq. 28, multiplying through by ϕ_n, and integrating over $0 \le \xi \le 1$, we find

$$\ddot{q}_n + \Omega_n^2 q_n + \epsilon\sum_j a_{nj}\dot{q}_j = 0, \tag{32}$$

where

$$\epsilon = 2\beta u$$

and

$$a_{nj} = \int_0^1 \phi_n \frac{\partial\phi_j}{\partial\xi}\, d\xi. \tag{33}$$

Note that Eq. 32 is coupled through the coefficient a_{nj}, which represents the effect of the Coriolis force term. From Eq. 33, it follows that

$$a_{nj} + a_{jn} = \phi_n(1)\phi_j(1) + \phi_n(0)\phi_j(0). \tag{34}$$

Since ϕ_n and ϕ_j satisfy the boundary conditions, the right-hand side of Eq. 34 is equal to zero. Accordingly, a_{nj} is skew-symmetric; i.e.,

$$a_{nj} = -a_{jn}. \tag{35}$$

This property can also be shown by carrying out the integration of Eq. 33. Integrating Eq. 33 gives

$$\left.\begin{aligned} a_{nj} &= 0, && n = j, \\ &= j\left[\frac{1-(-1)^{n-j}}{n-j} + \frac{1-(-1)^{n+j}}{n+j}\right], && n \neq j; \end{aligned}\right\} \tag{36}$$

a_{nj} is obviously skew-symmetric.

The system including the Coriolis-force term is characterized by Eq. 32, which consists of an infinite number of differential equations. However, typically, only a finite number of equations are selected from case to case, according to the desired accuracy. To find the natural frequencies, let

$$q_n = \alpha_n \exp(i\omega\tau). \tag{37}$$

Substitution of Eq. 37 into Eq. 32 leads to the algebraic equations

$$(\Omega_n^2 - \omega^2)\,\alpha_n + i\epsilon\omega \sum_j a_{nj}\alpha_j = 0, \qquad n,j = 1, 2, 3, \ldots . \tag{38}$$

A necessary condition for the existence of a nontrivial solution is

$$\left|(\Omega_n^2 - \omega^2)\delta_{nj} + i\epsilon\omega a_{nj}\right| = 0. \tag{39}$$

The natural frequencies ω_1, ω_2, ω_3, ..., are found by solving this characteristic equation. The displacement of the nth mode is

$$w_n = \sum_j \alpha_j^{(n)}\phi_j(\xi)\exp(i\omega_n\tau), \tag{40}$$

where $\alpha_j^{(n)}$ is the solution of Eqs. 38 corresponding to the natural frequency ω_n. The mode shape may be written in real form as

$$w_n = V_n(\xi)\cos\left[\omega_n\tau + \psi_n(\xi) + \theta_n\right], \tag{41}$$

where

$$\left.\begin{aligned} V_n &= \left\{\left[\mathrm{Re}\sum_j \alpha_j^{(n)}\phi_j(\xi)\right]^2 + \left[\mathrm{Im}\alpha_j^{(n)}\phi_j(\xi)\right]^2\right\}^{1/2}, \\ \psi_n &= \tan^{-1}\frac{\mathrm{Im}\sum_j \alpha_j^{(n)}\phi_j}{\mathrm{Re}\sum_j \alpha_j^{(n)}\phi_j}, \end{aligned}\right\} \tag{42}$$

and θ_n is an arbitrary phase constant. The phase variation along the tube is contained in ψ_n, which is much more complicated and cannot be described by a phase-propagating velocity such as that in the media without flexural rigidity.

The natural frequency of the fundamental mode has been computed taking two-mode, three-mode, and four-mode approximations (as listed in Table I) for comparison. This method is very convenient and provides sufficiently accurate results, especially when u is small compared with its critical value. A two-mode approximation will give sufficient accuracy for the first mode.

TABLE I. Fundamental Natural Frequency, ω_1/ν_1, Computed from Two-mode, Three-mode, and Four-mode Approximation

β:	0.2			0.6			1.0		
Modes: u	2	3	4	2	3	4	2	3	4
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.2	0.99789	0.99789	0.99789	0.99727	0.99727	0.99725	0.99603	0.99603	0.99597
0.5	0.98677	0.98677	0.98675	0.98296	0.98295	0.98282	0.97554	0.97543	0.97509
1.0	0.94614	0.94614	0.94608	0.93195	0.93176	0.93131	0.90598	0.90475	0.90376
1.5	0.87483	0.87482	0.87470	0.84649	0.84581	0.84499	0.79888	0.79520	0.79378
2.0	0.76525	0.76523	0.76505	0.72325	0.72200	0.72089	0.65896	0.65333	0.65170
2.5	0.55837	0.59835	0.59813	0.54954	0.54828	0.54701	0.48145	0.47674	0.47495
3.0	0.29176	0.29176	0.29160	0.25902	0.25876	0.25784	0.21739	0.21657	0.21517
π	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Figure 6 presents the variation of fundamental natural frequency as a function of the fluid transport velocity (u) when the axial force Γ is taken as zero. It is apparent that the flow does have a tendency to reduce the natural frequency; this is attributed to the centrifugal force and the Coriolis force. The centrifugal force plays the same role as that of the axial compression, which is known to reduce the natural frequencies, and it is much more important than the Coriolis force in reducing the frequencies. However, the latter produces a phase difference along the tube in addition to its small effect on natural frequencies.

It is observed in Fig. 6 that the critical fluid transport velocity ($u_{critical} = \pi$) is independent of β; alternatively, the Coriolis force does not affect the instability. If one is interested in the stability analysis, the Coriolis-force term may be dropped. The physical reasoning is discussed in Section VII.

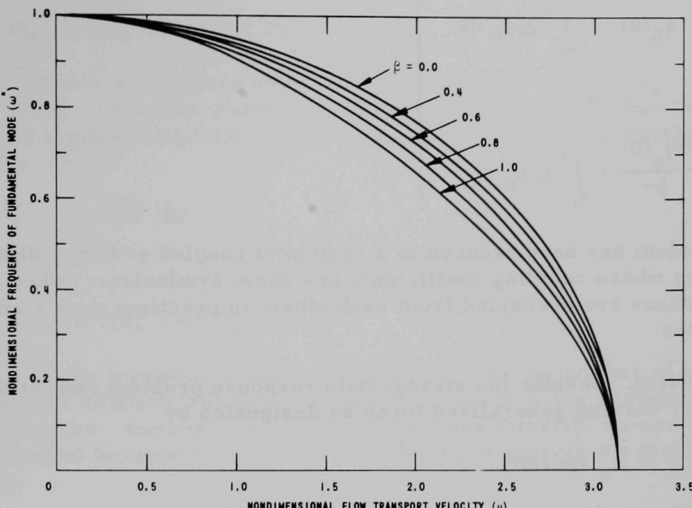

Fig. 6. Fundamental Natural Frequency vs Nondimensional Fluid Transport Velocity. ANL Neg. No. 113-3639.

The dynamic behavior is essentially the same as that discussed in Section III.B. The only difference is that the tube with rigidity is a dispersive medium. The component frequencies of any bending wave will propagate with different velocities, and dispersion will occur.

V. FORCED VIBRATION

For forced vibration, a solution of the form of Eq. 29 is assumed, and after substitution into Eq. 9 and utilization of simple transforms, we arrive at the ordinary differential equations

$$\ddot{q}_n + 2\zeta_n\dot{q}_n + \epsilon\sum_j a_{nj}\dot{q}_j + \Omega_n^2 q_n = Q_n, \tag{43}$$

where

$$\left.\begin{aligned} \zeta_n &= \tfrac{1}{2}(u_n^4\alpha + \delta), \\ Q_n &= \int_0^1 \left[Q - (u^2 - \Gamma)\frac{\partial^2 w_0}{\partial \xi^2}\right]\phi_n\, d\xi, \end{aligned}\right\} \tag{44}$$

and ϵ and a_{nj} have been defined in Eqs. 33. The initial conditions for these equations are

$$\left.\begin{aligned} q_n(0) &= \int_0^1 g_0 \phi_n \, d\xi \\ &\text{and} \\ \frac{\partial q_n(0)}{\partial \tau} &= \int_0^1 h_0 \phi_n \, d\xi. \end{aligned}\right\} \qquad (45)$$

The problem has been reduced to a system of coupled ordinary differential equations whose coupling coefficients are skew-symmetric. When $\epsilon = 0$, the equations are decoupled from each other; in practice, ϵ is a small parameter.

First, consider the steady-state response problem and let the harmonically varying generalized force be designated by

$$Q_n = \alpha_n \sin(\omega\tau). \qquad (46)$$

The steady-state solutions of Eqs. 43 are assumed to be

$$q_n = a_n \sin(\omega\tau) + b_n \cos(\omega\tau). \qquad (47)$$

Substituting of Eqs. 46 and 47 into Eq. 43, and setting the coefficients of $\sin(\omega\tau)$ and $\cos(\omega\tau)$ of the resulting equation equal to zero, we obtain

$$\left.\begin{aligned} (\Omega_n^2 - \omega^2)\, a_n - 2\zeta_n \omega b_n - \epsilon\omega \sum_j a_{nj} b_j &= \alpha_n \\ &\text{and} \\ (\Omega_n^2 - \omega^2)\, b_n + 2\zeta_n \omega a_n + \epsilon\omega \sum_j a_{nj} a_j &= 0. \end{aligned}\right\} \qquad (48)$$

Employing matrix notation, we write Eqs. 48 as

$$\left.\begin{aligned} \bar{\bar{\Omega}}\vec{a} - \bar{\bar{D}}\vec{b} &= \vec{\alpha} \\ &\text{and} \\ \bar{\bar{\Omega}}\vec{b} + \bar{\bar{D}}\vec{a} &= 0, \end{aligned}\right\} \qquad (49)$$

where $\bar{\bar{\Omega}}$ and $\bar{\bar{D}}$ are matrices whose elements are

$$\left.\begin{aligned} \Omega_{nj} &= (\Omega_n^2 - \omega^2)\,\delta_{nj} \\ &\text{and} \\ D_{nj} &= (2\zeta_n \delta_{nj} + \epsilon a_{nj})\,\omega, \end{aligned}\right\} \tag{50}$$

and $\vec{a}$, $\vec{b}$, and $\vec{\alpha}$ are column matrices whose components are a_n, b_n, and α_n, respectively. Thus, the steady-state problem is reduced to that of solving a system of algebraic equations. The values for $\vec{a}$ and $\vec{b}$ are easily obtained from Eqs. 49:

$$\left.\begin{aligned} \vec{a} &= (\bar{\bar{\Omega}} + \bar{\bar{D}}\bar{\bar{\Omega}}^{-1}\bar{\bar{D}})^{-1}\,\vec{\alpha} \\ &\text{and} \\ \vec{b} &= -\bar{\bar{\Omega}}^{-1}\bar{\bar{D}}(\bar{\bar{\Omega}} + \bar{\bar{D}}\bar{\bar{\Omega}}^{-1}\bar{\bar{D}})^{-1}\,\vec{\alpha}. \end{aligned}\right\} \tag{51}$$

For the transient response problem, if a large number of modes are included, the solution will be quite complicated due to the presence of the coupling terms. Instead of solving the equations directly, we use a perturbation method borrowed from nonlinear theory to analyze the transient problem.

For sufficiently small value of ϵ, we may develop the unknown function q_n of Eqs. 43 in power series with respect to ϵ:

$$q_n = q_n^{(0)} + \epsilon q_n^{(1)} + \epsilon^2 q_n^{(2)} + \dots \,. \tag{52}$$

Substituting the expansion into Eqs. 43 and 45 and collecting the coefficients of the like powers of ϵ yields the following sequence of uncoupled differential equations, which can be solved in order:

$$\underline{\epsilon^0:}\quad \ddot{q}_n^{(0)} + 2\zeta_n \dot{q}_n^{(0)} + \Omega_n^2 q_n^{(0)} = Q_n;\quad q_n^{(0)}(0) = q_n(0),\ \dot{q}_n^{(0)}(0) = \dot{q}_n(0). \tag{53a}$$

$$\underline{\epsilon^1:}\quad \ddot{q}_n^{(1)} + 2\zeta_n \dot{q}_n^{(1)} + \Omega_n^2 q_n^{(1)} = -\sum_j a_{nj}\dot{q}_j^{(0)};\quad q_n^{(1)}(0) = 0,\ \dot{q}_n^{(1)}(0) = 0. \tag{53b}$$

$$\underline{\epsilon^2:}\quad \ddot{q}_n^{(2)} + 2\zeta_n \dot{q}_n^{(2)} + \Omega_n^2 q_n^{(2)} = -\sum_j a_{nj}\dot{q}_j^{(1)};\quad q_n^{(2)}(0) = 0,\ \dot{q}_n^{(2)}(0) = 0. \tag{53c}$$

The zero equations are those of the forced motions without Coriolis force; the corrections $q_n^{(1)}$ are the forced motions excited by the Coriolis force associated with $q_n^{(0)}$, and so on. By this technique, the equations are uncoupled and easily solved. The solutions to Eqs. 53a take three possible forms, depending on the size of the damping coefficient.

Case I

If $\zeta_n^2 < \Omega_n^2$,

$$q_n^{(0)} = q_n^{(0)} \exp(-\zeta_n \tau) \cos \left(\sqrt{\Omega_n^2 - \zeta_n^2}\, \tau\right) + \frac{\dot{q}_n(0) + \zeta_n q_n(0)}{\sqrt{\Omega_n^2 - \zeta_n^2}}$$

$$\cdot \exp(-\zeta_n \tau) \sin \left(\sqrt{\Omega_n^2 - \zeta_n^2}\, \tau\right) + \frac{1}{\sqrt{\Omega_n^2 - \zeta_n^2}} Q_n(\tau)^* \exp(-\zeta_n \tau) \sin \left(\sqrt{\Omega_n^2 - \zeta_n^2}\, \tau\right). \tag{54a}$$

Case II

If $\zeta_n^2 = \Omega_n^2$,

$$q_n^{(0)} = q_n(0) \exp(-\zeta_n \tau) + [\dot{q}_n(0) + \zeta_n q_n(0)]\tau \exp(-\zeta_n \tau) + Q_n(\tau)^* \tau \exp(-\zeta_n \tau). \tag{54b}$$

Case III

If $\zeta_n^2 > \Omega_n^2$,

$$q_n^{(0)} = q_n(0) \exp(-\zeta_n \tau) \cosh \left(\sqrt{\zeta_n^2 - \Omega_n^2}\, \tau\right) + \frac{\dot{q}_n(0) + \zeta_n q_n(0)}{\sqrt{\zeta_n^2 - \Omega_n^2}}$$

$$\cdot \exp(-\zeta_n \tau) \sinh \left(\sqrt{\zeta_n^2 - \Omega_n^2}\, \tau\right) + \frac{1}{\sqrt{\zeta_n^2 - \Omega_n^2}} Q_n(\tau)^* \exp(-\zeta_n \tau) \sinh \left(\sqrt{\zeta_n^2 - \Omega_n^2}\, \tau\right). \tag{54c}$$

In the above equations,

$$Q(\tau)^* p(\tau) = \int_0^\tau Q(\tau') p(\tau - \tau')\, d\tau'. \tag{54d}$$

By substituting the results into first-order equations, we obtain the solutions for the next order in similar manner.

VI. PARAMETRIC RESPONSE

In this section, the problem considered is the dynamic stability of the tube under pulsating flow. The velocity varies with time and is represented in the form

$$u(\tau) = u_0[1 + \mu \cos(\omega \tau)], \tag{55}$$

where u_0 and μ are constants. The fluctuation of flow velocity can result from a pumping operation or the unsteady component in a two-phase flow.

The equation of motion takes the form

$$\frac{\partial^4 w}{\partial \xi^4} + \alpha \frac{\partial^5 w}{\partial \xi^4 \partial \tau} + u_0^2[1 + \mu \cos(\omega\tau)]^2 \frac{\partial^2 w}{\partial \xi^2} + 2\beta u_0[1 + \mu \cos(\omega\tau)] \frac{\partial^2 \omega}{\partial \xi \partial \tau}$$

$$- \beta u_0 \mu \omega \sin(\omega\tau) \frac{\partial w}{\partial \xi} + \delta \frac{\partial w}{\partial \tau} + \frac{\partial^2 w}{\partial \tau^2} = Q. \tag{56}$$

The solution of Eq. 56 is sought in the form of Eqs. 29 and 31a. Applying Galerkin's method, we determine a system of ordinary differential equations for the unknown functions of the time coordinates $q_n(\tau)$:

$$\ddot{q}_n + 2\zeta_n \nu_n \dot{q}_n + \epsilon[1 + \mu \cos(\omega\tau)] \sum_j a_{nj} \dot{q}_j - \epsilon_1 \omega \sin(\omega\tau) \sum_j a_{nj} q_j$$

$$+ \nu_n^2 \left\{1 - [1 + \mu \cos(\omega\tau)]^2 r_n^2\right\} q_n = Q_n, \tag{57}$$

where

$$\left.\begin{aligned} \epsilon &= 2\beta u_0, \\ \nu_n &= n^2\pi^2, \\ \zeta_n &= \frac{\nu_n}{2}\alpha + \frac{1}{2\nu_n}\delta, \\ r_n &= \frac{u_0}{u_n}, \\ \epsilon_1 &= \beta u_0 \mu, \\ Q_n &= \int_0^1 Q\phi_n \, d\xi, \end{aligned}\right\} \tag{58}$$

and the other notations have been defined before. Since only the stability problem is discussed here, Q_n is taken as zero, or more precisely, there is no external force and no initial curvature. Thus, Eqs. 57 may be written

$$\frac{1}{\nu_n^2}\ddot{q}_n + \left[1 - \rho_1 r_n^2 - \rho_2 r_n^2 \cos(\omega\tau) - \rho_n r_n^2 \cos(2\omega\tau)\right] q_n - \epsilon_1 \omega \sin(\omega\tau) \sum_j \frac{a_{nj}}{\nu_n^2} q_j$$

$$+ 2\frac{\zeta_n}{\nu_n}\dot{q}_n + \epsilon[1 + \mu \cos(\omega\tau)] \sum_j \frac{a_{nj}}{\nu_n^2} \dot{q}_j = 0, \tag{59}$$

where

$$
\left.
\begin{aligned}
\rho_1 &= 1 + \tfrac{1}{2}\mu^2, \\
\rho_2 &= 2\mu, \\
\text{and} \\
\rho_3 &= \tfrac{1}{2}\mu^2.
\end{aligned}
\right\} \tag{60}
$$

The equations may be further condensed into one vector equation:

$$
\begin{aligned}
&\bar{\bar{\bar{M}}}\ddot{\vec{q}} + \left[\bar{\bar{E}} - \rho_1\bar{\bar{B}} - \rho_2 \cos(\omega\tau)\bar{\bar{B}} - \rho_3 \cos(2\omega\tau)\bar{\bar{B}}\right]\vec{q} - \epsilon_1\omega \sin(\omega\tau)\bar{\bar{A}}\vec{q} \\
&\quad + 2\epsilon[1 + \mu\cos(\omega\tau)]\bar{\bar{A}}\dot{\vec{q}} + 2\bar{\bar{D}}\dot{\vec{q}} = 0,
\end{aligned} \tag{61}
$$

where $\bar{\bar{\bar{M}}}$, $\bar{\bar{E}}$, $\bar{\bar{B}}$, and $\bar{\bar{D}}$ are diagonal matrices with the elements M_{nj}, E_{nj}, B_{nj}, and D_{nj}, respectively; and

$$
\left.
\begin{aligned}
M_{nj} &= \frac{1}{\nu_n^2}\delta_{nj}, \\
E_{nj} &= \delta_{nj}, \\
B_{nj} &= r_n^2\delta_{nj}, \\
\text{and} \\
D_{nj} &= \frac{1}{\nu_n}\zeta_n\delta_{nj}.
\end{aligned}
\right\} \tag{62}
$$

Matrix $\bar{\bar{A}}$ is antisymmetric and is the coupling term resulting from the Coriolis force; its element A_{nj} is

$$
A_{nj} = \frac{1}{\nu_n^2}a_{nj}. \tag{63}
$$

Equations 61 are the coupled differential equations of the Mathieu-Hill type with multiharmonic coefficients. A brief outline of the general theory of stability of these equations is given here; it is patterned closely after the discussion by Bolotin.[9]

The solution of this system, which corresponds to the boundaries of the instability region and has a period of 2P ($P = 2\pi/\omega$), is sought in the form

$$\vec{q} = \sum_{k=1,3,5,\ldots} \left(\vec{a}_k \sin\frac{k\omega\tau}{2} + \vec{b}_k \cos\frac{k\omega\tau}{2}\right). \tag{64}$$

On substituting Eq. 64 into Eq. 61, multiplying out all trigonometric functions, and equating to zero the coefficients of sin $(k\omega\tau/2)$ and cos $(k\omega\tau/2)$ with the same index k, we obtain the following infinite system of algebraic equations for the coefficients $\vec{a}_k$ and $\vec{b}_k$:

$$\left.\begin{aligned}
&-\frac{\rho_3}{2}\bar{\bar{B}}\vec{a}_5 - \tfrac{1}{2}(\rho_2-\rho_3)\,\bar{\bar{B}}\vec{a}_3 + \left(\bar{\bar{E}} - \vec{\rho}_1\bar{\bar{B}} + \frac{\rho_2}{2}\bar{\bar{B}} - \frac{\omega^2}{4}\bar{\bar{M}}\right)\vec{a}_1 \\
&\qquad - \omega\left(\epsilon\bar{\bar{A}} + \bar{\bar{D}}\right)\vec{b}_1 + 2\epsilon\mu\omega\bar{\bar{A}}\vec{b}_3 = 0; \\
&\mu\epsilon\omega\bar{\bar{A}}\vec{a}_3 + \omega\left(\epsilon\mu\bar{\bar{A}} + \bar{\bar{D}}\right)\vec{a}_1 + \left(\bar{\bar{E}} - \rho_1\bar{\bar{B}} - \tfrac{1}{2}\rho_2\bar{\bar{B}} - \frac{\omega^2}{4}\bar{\bar{M}}\right)\vec{b}_1 \\
&\qquad - \tfrac{1}{2}(\rho_2+\rho_3)\,\bar{\bar{B}}\vec{b}_3 - \tfrac{1}{2}\rho_3\bar{\bar{B}}b_5 = 0; \\
&-\tfrac{1}{2}\rho_3\bar{\bar{B}}\vec{a}_7 - \tfrac{1}{2}\rho_2\bar{\bar{B}}\vec{a}_5 + \left(\bar{\bar{E}} - \rho_1\bar{\bar{B}} - \frac{9}{4}\omega^2\bar{\bar{M}}\right)\vec{a}_3 - \tfrac{1}{2}(\rho_2-\rho_3)\,\vec{a}_1 - \mu\epsilon\omega\bar{\bar{A}}\vec{b}_1 \\
&\qquad - 3\omega(\epsilon\bar{\bar{A}} + \bar{\bar{D}})\,\vec{b}_3 - \mu\epsilon\omega\bar{\bar{A}}\vec{b}_5 = 0; \\
&2\epsilon\mu\bar{\bar{A}}\vec{a}_5 + 3\omega(\epsilon\bar{\bar{A}} + \bar{\bar{D}})\,\vec{a}_3 + \epsilon\mu\omega\bar{\bar{A}}\vec{a}_1 - \tfrac{1}{2}(\rho_2-\rho_3)\,\bar{\bar{B}}\vec{b}_1 + \left(\bar{\bar{E}} - \rho_1\bar{\bar{B}} - \frac{9}{4}\omega^2\bar{\bar{M}}\right)\vec{b}_3 \\
&\qquad - \tfrac{1}{2}\rho_2\bar{\bar{B}}\vec{b}_5 + \tfrac{1}{2}\rho_3\bar{\bar{B}}\vec{b}_7 = 0; \\
&\left(\bar{\bar{E}} - \rho_1\bar{\bar{B}} - \frac{k^2\omega^2}{4}\bar{\bar{M}}\right)\vec{a}_k - \tfrac{1}{2}\rho_2\bar{\bar{B}}\left(\vec{a}_{k+2} + \vec{a}_{k-2}\right) - \frac{\rho_3}{2}\bar{\bar{B}}\left(\vec{a}_{k+4} + \vec{a}_{k-4}\right) \\
&\qquad - k\omega\left(\epsilon\bar{\bar{A}} + \bar{\bar{D}}\right)\vec{b}_k - \epsilon\mu\bar{\bar{A}}\left(\frac{k+3}{2}\vec{b}_{k+2} + \frac{k-3}{2}\vec{b}_{k-2}\right) = 0; \\
&\text{and} \\
&\left(\bar{\bar{E}} - \rho_1\bar{\bar{B}} - \frac{k^2\omega^2}{4}\bar{\bar{M}}\right)\vec{b}_k - \tfrac{1}{2}\rho_2\bar{\bar{B}}\left(\vec{b}_{k+2} + \vec{b}_{k-2}\right) - \tfrac{1}{2}\rho_3\bar{\bar{B}}\left(\vec{b}_{k+4} + \vec{b}_{k-4}\right) \\
&\qquad + k\omega\left(\epsilon\bar{\bar{A}} + \bar{\bar{D}}\right)\vec{a}_k + \epsilon\mu\bar{\bar{A}}\left(\frac{k+1}{2}\vec{a}_{k+2} + \frac{k-1}{2}\vec{b}_{k-2}\right) = 0. \\
&\qquad k = 5, 7, 9, \ldots .
\end{aligned}\right\} \tag{65}$$

Equations 65 are equivalent to the ordinary differential Eqs. 61; they may be condensed in matrix form as follows:

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \bar{\bar{E}}_{22} & \bar{\bar{E}}_{21} & \bar{\bar{F}}_{21} & \bar{\bar{F}}_{22} & \cdots \\ \cdots & \bar{\bar{E}}_{12} & \bar{\bar{E}}_{11} & \bar{\bar{F}}_{11} & \bar{\bar{F}}_{12} & \cdots \\ \cdots & \bar{\bar{G}}_{12} & \bar{\bar{G}}_{11} & \bar{\bar{H}}_{11} & \bar{\bar{H}}_{12} & \cdots \\ \cdots & \bar{\bar{G}}_{22} & \bar{\bar{G}}_{21} & \bar{\bar{H}}_{21} & \bar{\bar{H}}_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vec{a}_3 \\ \vec{a}_1 \\ \vec{b}_1 \\ \vec{b}_3 \\ \vdots \end{bmatrix} = 0, \tag{66}$$

where

$$\left.\begin{aligned} \bar{\bar{E}}_{11} &= \bar{\bar{E}} - \rho_1 \bar{\bar{B}} + \frac{\rho_2}{2} \bar{\bar{B}} - \frac{\omega^2}{4} \overline{\overline{M}}, \\ \bar{\bar{F}}_{11} &= -\epsilon\omega \bar{\bar{A}} - \omega \bar{\bar{D}}, \\ \bar{\bar{G}}_{11} &= -\epsilon\omega \bar{\bar{A}} + \omega \bar{\bar{D}}, \\ \bar{\bar{H}}_{11} &= \bar{\bar{E}} - (\rho_1 + \tfrac{1}{2}\rho_2)\, \bar{\bar{B}} - \frac{\omega^2}{4} \bar{\bar{D}}, \\ &\vdots \end{aligned}\right\} \tag{67}$$

The boundary equation of the instability regions is obtained by setting the determinant of the coefficient of $\vec{a}_k$ and $\vec{b}_k$ equal to zero; i.e.,

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \bar{\bar{E}}_{22} & \bar{\bar{E}}_{21} & \bar{\bar{F}}_{21} & \bar{\bar{F}}_{22} & \cdots \\ \cdots & \bar{\bar{E}}_{12} & \bar{\bar{E}}_{11} & \bar{\bar{F}}_{11} & \bar{\bar{F}}_{12} & \cdots \\ \cdots & \bar{\bar{G}}_{12} & \bar{\bar{G}}_{11} & \bar{\bar{H}}_{11} & \bar{\bar{H}}_{12} & \cdots \\ \cdots & \bar{\bar{G}}_{22} & \bar{\bar{G}}_{21} & \bar{\bar{H}}_{21} & \bar{\bar{H}}_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{68}$$

The determinant is infinite. Bolotin shows that this infinite determinant belongs to the class of normal determinant implying absolute convergence. In view of this property, the boundary of the principal region of instability may be located approximately by equating to zero the following determinant:

$$\begin{vmatrix} \bar{\bar{E}}_{11} & \bar{\bar{F}}_{11} \\ \bar{\bar{G}}_{11} & \bar{\bar{H}}_{11} \end{vmatrix}. \tag{69}$$

The principal region of instability is expressed in the form (see Appendix B)

$$\sqrt{\chi_2} \le \frac{\omega}{2\nu_1} \le \sqrt{\chi_1}, \tag{70}$$

where χ_1 and χ_2 depend on the parameters β, u, and μ. If the damping is not taken into account, the solution of Eqs. 69 gives the following approximate formulas for the boundaries:

$$\left.\begin{aligned} \tfrac{1}{4}\chi_1 &= 1 - (1-\mu+\tfrac{1}{2}\mu^2)\, r_1^2 \\ &\quad - \frac{\frac{256}{9}\beta^2 u^2[1-(1-\mu+\tfrac{1}{2}\mu^2)\, r_1^2]}{\nu_2^2[1-(1+\mu+\tfrac{1}{2}\mu^2)\, r_2^2] - \nu_1^2[1-(1-\mu+\tfrac{1}{2}\mu^2)\, r_1^2]}, \\ &\text{and} \\ \tfrac{1}{4}\chi_2 &= 1 - (1+\mu+\tfrac{1}{2}\mu^2)\, r_1^2 \\ &\quad - \frac{\frac{256}{9}\beta^2 u^2[1-(1+\mu+\tfrac{1}{2}\mu^2)\, r_1^2]}{\nu_2^2[1-(1-\mu+\tfrac{1}{2}\mu^2)\, r_2^2] - \nu_1^2[1-(1+\mu+\tfrac{1}{2}\mu^2)\, r_1^2]}. \end{aligned}\right\} \tag{71}$$

If the index k in Eqs. 64 is evaluated, 0, 2, 4, ..., the function $\vec{q}$ will be the solution corresponding to the boundaries of the instability having period P, and the calculations are carried out in the same manner as before. The solutions obtained determine the boundaries of the second instability region, which is expressed by

$$\sqrt{\tau_2} \le \frac{\omega}{\nu_1} \le \sqrt{\tau_1}. \tag{72}$$

Here, τ_1 and τ_2 also depend on the same parameters. Without considering the damping, the approximate formulas determining the boundaries of the second instability region are:

$$\left.\begin{aligned} \tau_1 &= 1 - (1+\tfrac{1}{4}\mu^2)\, r_1^2 \\ &\quad - \frac{\frac{256}{9}\beta^2 u^2[1-(1+\tfrac{1}{2}\mu^2)\, r_2^2][1-(1+\tfrac{1}{4}\mu^2)\, r_1^2]}{[1-(1+\tfrac{1}{2}\mu^2)\, r_2^2][\nu_2^2-(1+\tfrac{3}{4}\mu^2)\, r_2^2\nu_2^2+(1+\tfrac{1}{4}\mu^2)\, r_1^2\nu_1^2-\nu_1^2]-2\mu^2 r_2^4\nu_2^2}, \end{aligned}\right\} \tag{73}$$

(73) Contd.

and

$$\left.\begin{aligned}\tau_2 &= 1 - (1+\tfrac{3}{4}\mu^2)\ r_1^2 - \frac{2\mu^2 r_1^4}{1 - (1+\tfrac{1}{2}\mu^2)\ r_1^2} \\ &- \frac{\frac{256}{9}\beta^2 u^2\{[1-(1+\tfrac{1}{2}\mu^2)\ r_1^2][1-(1+\tfrac{3}{4}\mu^2)\ r_1^2] - 2\mu^2 r_1^4\}}{[1-(1+\tfrac{1}{2}\mu^2)\ r_1^2][\nu_2^2 - (1+\tfrac{1}{4}\mu^2)\ r_2^2\nu_2^2 + (1+\tfrac{3}{4}\mu^2)\ r_1^2\nu_1^2 - \nu_1^2] - 2\mu^2 r_1^4\nu_1^2}.\end{aligned}\right\}\begin{matrix}\text{Contd.}\\ (73)\end{matrix}$$

Results of calculations using these formulas and several sets of parameters are shown in Figs. 7 and 8. If a point occurs in the noncross-hatched region, the initial straight form of the tube is dynamically stable.

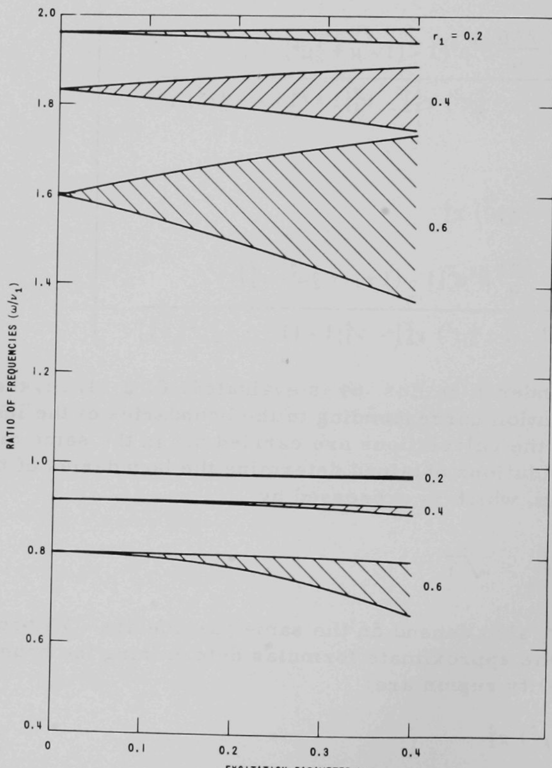

Fig. 7. Stability Diagram for β = 0.2.
ANL Neg. No. 113-3631 Rev. 1.

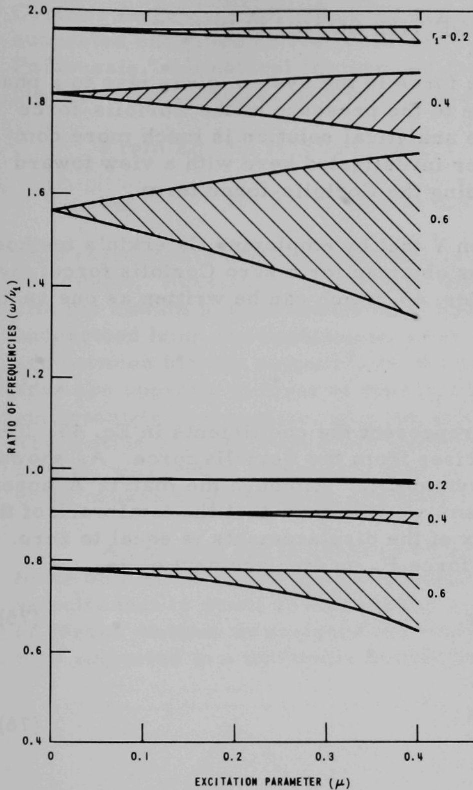

Fig. 8. Stability Diagram for β = 0.8.
ANL Neg. No. 113-3630 Rev. 1.

However, if it occurs in the crosshatched region, the tube will be dynamically unstable, and any initial deviation from the straight form will increase without bound.

A few conclusions can be drawn from the results:

1. The second instability region is much narrower than the principal region.

2. The width of the instability regions depends on the velocity component. As r_1 increases, the parametric resonance frequency decreases and the width of the instability region increases.

3. The effect of the Coriolis force is to lower slightly the instability region.

4. The instabilities at constant flow velocity need not concern reactor designers, because they occur at high transport velocities that are not likely to occur in reactor channels; however, parametric resonance can occur at lower flow velocities.

Because of the importance of the nonlinear effects, the linear theory is unable to predict the amplitude of oscillation under unstable conditions. Therefore, nonlinear theory should be used in determining the amplitude response in the instability regions. However, from a practical point of view, the important consideration is to avoid the onset of instability; this can be achieved on the basis of linear theory. As long as the system properties are known, the instability can be checked, using the method and results presented here.

VII. SIGNIFICANCE OF THE CORIOLIS FORCE

The existence of a Coriolis force in the system gives rise to a phase difference and mode coupling. Due to the presence of the Coriolis-force term in the equation of motion, the analytical solution is much more complicated. Its significance is further investigated here with a view toward assessing the importance of including the Coriolis-force term.

It has been shown in Section V that by employing Galerkin's method and using the mode-shape functions obtained for a zero Coriolis force, the equation of motion is reduced to Eqs. 43, which can be written as one vector equation:

$$\ddot{\vec{q}} + 2\bar{\bar{D}}\dot{\vec{q}} + \epsilon\bar{\bar{A}}\dot{\vec{q}} + \bar{\bar{\Omega}}\vec{q} = \vec{Q}, \tag{74}$$

where the matrices $\bar{\bar{D}}$, $\bar{\bar{A}}$, and $\bar{\bar{\Omega}}$ represent the coefficients in Eq. 43. The third term on the left-hand side arises from the Coriolis force. As shown in Eq. 35, the matrix $\bar{\bar{A}}$ is skew-symmetric. Although the matrix $\bar{\bar{A}}$ appears as an energy-dissipation mechanism, we will show that the total work of the forces corresponding to the matrix of the displacements is equal to zero. The virtual work of a generalized force F_n on displacement q_n is

$$dW = \sum_n F_n dq_n = \sum_n F_n \dot{q}_n \, d\tau. \tag{75}$$

If

$$F_n = -\epsilon \sum a_{nj}\dot{q}_j \tag{76}$$

is substituted into Eq. 75,

$$dW = -\epsilon \sum_n \sum_j a_{nj}\dot{q}_n\dot{q}_j \, d\tau. \tag{77a}$$

Using the result given by Eq. 35, one obtains

$$dW = 0. \tag{77b}$$

Thus the Coriolis force does not dissipate or supply any energy; i.e., it is not a resistant force or an energy source. This implies that during vibration the energy transfer between fluid and tube due to the Coriolis force is zero at any instant.

The skew-symmetry of the matrix $\bar{\bar{A}}$ is associated with the boundary conditions. Recalling Eq. 34, we know that as long as there is no displacement at the ends, the matrix is skew-symmetric and the Coriolis force will not contribute damping to the system. Consequently, the natural frequencies obtained from Eq. 39 are always real and positive in the subcritical region when the fluid transport velocity is less than its critical value. Furthermore, the tube and fluid comprise a conservative system; therefore, the

Coriolis force does not affect the instability. However, if there is an unsupported end, such as the cantilever tube discussed by Gregory and Paidoussis,[6] the natural frequencies in the subcritical region are complex, and the Coriolis force may have a destabilizing effect on the system.

From Eq. 36, it follows that

$$a_{nj} = 0 \quad \text{for } n,j = \text{even or } n,j = \text{odd}. \tag{78}$$

This relationship implies that there is no coupling between even modes or odd modes. Physically, this means that the motion in odd modes will excite the motion in even modes only, and vice versa. This is most easily understood from the fundamental mode. The displacement is a symmetric arc between the two supports, but the angular velocities of the left half span are opposite to those of the right half span; i.e., the Coriolis force is antisymmetric; therefore, only the antisymmetric modes will be excited. The motion of these modes will produce a symmetric Coriolis force which, in turn, will induce symmetric motion. This is the reason why the Coriolis force causes mode coupling and phase difference.

As pointed out in Sections IV and VI, the influence of the Coriolis force on free vibration and parametric response is small for a transport velocity that is small relative to the critical value. To observe the effect on forced motion, we analyzed the transient response of a simply supported tube subjected to a uniformly distributed step-function loading. With the method suggested in Section V, the displacements at a quarter point are shown in Fig. 9 for β = 0.8 and u = 0.5. Here, the dotted line is the solution neglecting the Coriolis force term (zero-order perturbation solution); the solid line is for the case including the Coriolis force (first-order perturbation solution). It is clear that the effect of the Coriolis force on the transient response is negligible.

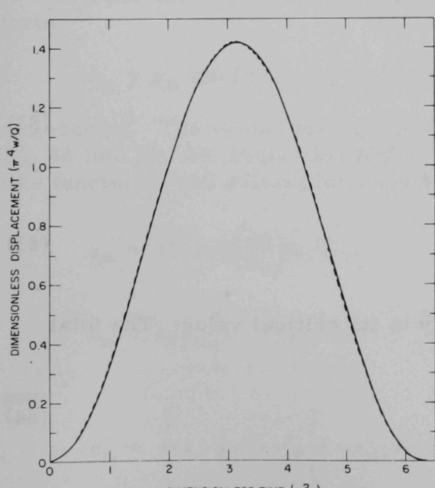

Fig. 9. Transient Response Curves for the Tube Displacements due to Step-function Excitation. ANL Neg. No. 113-3635 Rev. 1.

From an energy consideration, free vibration, forced motion, and parametric response, we may conclude that neglecting the Coriolis force in the study of the transverse vibration of a supported tube will not cause a large discrepancy over the range of fluid transport velocities important in practical reactor design.

VIII. INFLUENCE OF INITIAL CURVATURE

A. Steady Flow

If a tube containing static fluid is subjected to the action of a lateral load, a small initial curvature of the tube has no effect on the bending stress. However, if the fluid is dynamic, the deflection and bending stress will be substantially influenced by initial curvature.

Consider the static case. The initial shape of the axis of the tube is given by a series

$$w_0 = \sum_n f_n \sin(n\pi\xi), \tag{79}$$

where f_n is Fourier-expansion coefficient. The tube conveying steady flow is subjected to the action of a longitudinal compressive force u^2 (dimensionless). An additional deflection w_1 will be produced so that the final deflection is

$$w = w_1 + w_0. \tag{80}$$

The static deflection w_1 is determined from Eq. 9 by dropping the time-dependent terms (further assuming $Q = \Gamma = 0$); this yields

$$\frac{\partial^4 w_1}{\partial \xi^4} + u^2 \frac{\partial^2 w_1}{\partial \xi^2} = -u^2 \frac{\partial^2 w_0}{\partial \xi^2}, \tag{81}$$

the solution of which is

$$w_1 = \sum_n g_n \sin(n\pi\xi), \tag{82}$$

where

$$g_n = \frac{r_n^2}{1 - r_n^2} f_n \tag{83}$$

and r_n is the ratio of transport velocity to its critical value. The total deflection is

$$w = \sum_n \frac{f_n}{1 - r_n^2} \sin(n\pi\xi). \tag{84}$$

It is easily seen from Eq. 84 that the initial deflection corresponding to the nth mode is magnified in the ratio $1/(1 - r_n^2)$. As the transport velocity reaches its critical value, the deflection becomes infinite and buckling occurs; this is consistent with the result obtained from the dynamic analysis.

The dynamic effect of initial curvature under steady-flow conditions can be determined from Eq. 9 or Eqs. 43 and 44. The force induced by the initial curvature is time-independent; therefore its effect on the dynamic response of the tube is exactly the same as that of a distributed load of magnitude $u^2(\partial^2 w_0/\partial\xi^2)$ acting on the tube.

B. Pulsating Flow

To understand the influence of the initial curvature on a tube conveying pulsating flow, the following problem was analyzed:

A simply supported tube with an initial imperfection given by Eq. 79 conveys unsteady flow described in Eq. 55. It is assumed that the tube is free from external excitation and axial force. The problem is to determine the deflection of the tube. The basic differential equations have been formulated in Section VI, and are given by Eqs. 56-58 with the Coriolis force included. It has been shown that the effect of the Coriolis force is small when the transport velocity is small. As a preliminary study, the coupling effect due to the Coriolis force is neglected. If the initial imperfection w_0 is substituted into Eq. 58, then Eq. 57 reduces to

$$\ddot{q}_n + 2\zeta_n\nu_n\dot{q}_n + \nu_n^2[1 - \rho_1 r_n^2 - \rho_2 r_n^2 \cos(\omega\tau) - \rho_3 r_n^2 \cos(2\omega\tau)]\, q_n =$$
$$\nu_n^2[\rho_1 r_n^2 + \rho_2 r_n^2 \cos(\omega\tau) + \rho_3 r_n^2 \cos(2\omega\tau)]\, f_n. \tag{85}$$

Near the second-order resonance, the steady-state solution of the form

$$q_n = a_n \sin(\omega\tau) + b_n \cos(\omega\tau) + c_n \tag{86}$$

is assumed. The values for a_n, b_n, and c_n are determined by substituting Eq. 86 into Eq. 85, neglecting higher harmonics, equating coefficients of like functions, and solving the resulting equations. These operations yield

$$a_n = 4\zeta_n\mu r_n^2\left(\frac{\omega}{\nu_n}\right) f_n/\Delta_n,$$

$$b_n = 2\mu r_n\left[1 - r_n^2(1 + \tfrac{1}{4}\mu^2) - \left(\frac{\omega}{\nu_n}\right)^2\right] f_n/\Delta_n,$$

and

$$c_n = \left\{(1 + \tfrac{1}{2}\mu^2)\, r_n^2\left[1 - r_n^2(1 + \tfrac{1}{4}\mu^2) - \left(\frac{\omega}{\nu_n}\right)^2\right]^2 \right.$$
$$\left. + 2\mu^2 r_n^4\left[1 - r_n^2(1 + \tfrac{1}{4}\mu^2) - \left(\frac{\omega}{\nu_n}\right)^2\right]\right\} f_n/\Delta_n, \tag{87}$$

where

$$\Delta_n = [1 - (1 + \tfrac{1}{2}\mu^2)\, r_n^2]\left\{\left[1 - r_n^2(1 + \tfrac{1}{4}\mu^2) - \left(\frac{\omega}{\nu_n}\right)^2\right]^2 + 4\zeta_n^2\left(\frac{\omega}{\nu_n}\right)^2\right\} - 2\mu^2 r_n^4\left[1 - r_n^2(1 + \tfrac{1}{4}\mu^2) - \left(\frac{\omega}{\nu_n}\right)^2\right]. \tag{88}$$

The total steady-state amplitude is

$$w = \sum_n [f_n + c_n + a_n \sin(\omega\tau) + b_n \cos(\omega\tau)] \sin(n\pi\xi). \tag{89}$$

A numerical result is given for the curved tube whose unstressed deflection is assumed to be represented by one-half of a sine wave; i.e.,

$$w_0 = f_1 \sin(\pi\xi). \tag{90}$$

The steady-state solution is

$$w = [f_1 + c_1 + a_1 \sin(\omega\tau) + b_1 \cos(\omega\tau)] \sin(\pi\xi), \tag{91}$$

and the maximum steady-state amplitude is assumed to be

$$w_{max} = \left(f_1 + c_1 + \sqrt{a_1^2 + b_1^2}\right) \sin(\pi\xi). \tag{92}$$

The magnification factor is defined by

$$M.F. = \frac{w_{max}}{w_0} = 1 + \frac{1}{f_1}\left(c_1 + \sqrt{a_1^2 + b_1^2}\right). \tag{93}$$

The response curves are shown in Fig. 10. The amplitudes depend on the ratio of the frequencies and the parameters r_1, μ, and ζ_1. It is apparent that the initial imperfection is greatly magnified near the instability regions. Thus the curvature of the tube can produce large deflection under pulsating flow, especially when the fluid transport velocity is large.

The approximate solution obtained in this manner is valid near the second instability region. This is to demonstrate the effect of initial curvature under unsteady flow. If one is interested in the exact response problem, it is more realistic to use nonlinear theory,[8] which is not discussed here.

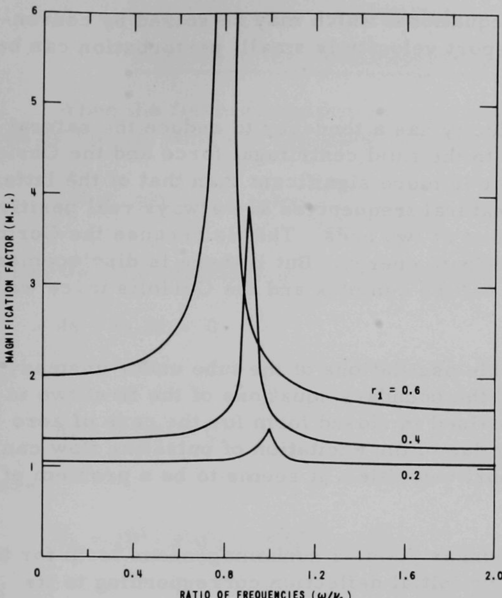

Fig. 10

Response Curves for Tube Displacements with $\mu = 0.2$, $\zeta_1 = 0.01$. ANL Neg. No. 113-3634 Rev. 1.

IX. SUMMARY AND CONCLUSIONS

The mathematical-model formulation in this report is appropriate for transverse-vibration studies of a tube conveying fluid with a relatively small transport velocity compared with its critical value. The effects of initial curvature, axial forces, viscous damping, and unsteady flow are included in the equation of motion. The peculiar feature is the presence of the mixed-derivative term. Mathematically, this term makes the boundary-value problems not self-adjoint and induces the coupling coefficients. Physically, it is of a Coriolis force origin. Due to its effect, the system does not possess the classical normal modes. Instead, the tube is characterized by phase difference and mode coupling. The mixed-derivative term also gives rise to two different phase velocities in the infinite media. Thus the propagations of initial disturbances and impressed force waves are quite different from those in fixed media. Although the mixed-derivative term produces a number of distinguishing features, it may be neglected without causing a large discrepancy for the case in which the transport velocity is small with respect to its critical value.

By employing the mode-shape functions obtained for zero Coriolis force, Galerkin's method is an efficient approach. The free vibration is reduced to an algebraic characteristic equation, which is readily solved and yields sufficiently accurate results. The forced vibration is transformed to a system of coupled ordinary equations, which may be solved by conventional methods; when the transport velocity is small, perturbation can be used.

The fluid transport velocity has a tendency to reduce the natural frequencies; this is attributed to the fluid centrifugal force and the Coriolis force. The effect of the former is more significant than that of the latter. In the subcritical region, the natural frequencies are always real positive for the tube without displacement at two ends. This is because the Coriolis force in this case does not dissipate energy. But if there is displacement at either end, the frequencies will be complex and the Coriolis force will damp the system.

The existence of unstable oscillations of the tube under unsteady flow has been established, and the boundary equations of the first two instability regions have been obtained in closed form for the case of zero damping. Since the instability due to the excitation of pulsating flow can occur at relatively low transport velocities, it seems to be a problem of importance.

The initial tube crookedness yields a nonhomogeneous term for the equation. In the steady flow, the initial deflection corresponding to its nth mode of buckling is magnified in the ratio of $1/(1 - r_n^2)$. But for the pulsating flow, the magnification factor becomes very large for certain frequency ratios. Initial curvature may induce large deflections if the transport velocity is large; it has no beneficial effect upon the vibrations and stability.

APPENDIX A

Solution of the Nonhomogeneous Wave Equation

When the flexural rigidity is neglected, the equation of motion is given by Eq. 12, which is repeated for convenience:

$$(\beta^2 U^2 - \bar{v}^2)\frac{\partial^2 y}{\partial x^2} + 2\beta^2 U \frac{\partial^2 y}{\partial x \partial t} + \frac{\partial^2 y}{\partial t^2} = R(x, t). \tag{A.1}$$

We see that the equations of the characteristics are

$$\left.\begin{aligned} &dx - \bar{v}_1\, dt = 0 \\ \text{and} \quad & \\ &dx + \bar{v}_2\, dt = 0, \end{aligned}\right\} \tag{A.2}$$

where

$$\left.\begin{aligned} &\bar{v}_1 = [\bar{v}^2 - \beta^2 U^2(1 - \beta^2)]^{1/2} + \beta^2 U \\ \text{and} \quad & \\ &\bar{v}_2 = [\bar{v} - \beta^2 U^2(1 - \beta^2)]^{1/2} - \beta^2 U. \end{aligned}\right\} \tag{A.3}$$

The sign of the quantity $\bar{v}^2 - \beta^2 U^2(1 - \beta^2)$ determines whether Eq. A.1 is hyperbolic or elliptic. For small velocities U, the equation is hyperbolic; for sufficiently high velocities, it is elliptic. For the particular velocity

$$U = \frac{\bar{v}}{\beta\sqrt{1 - \beta^2}}, \tag{A.4}$$

it becomes parabolic.

In the following analysis, we consider the case in which

$$U < \frac{\bar{v}}{\beta\sqrt{1 - \beta^2}}.$$

Assuming that the initial conditions are prescribed by

$$\left.\begin{aligned} &y = g(x) \quad \text{at } t = 0, \\ \text{and} \quad & \\ &\frac{\partial y}{\partial t} = h(x) \quad \text{at } t = 0, \end{aligned}\right\} \tag{A.5}$$

we are interested in the solution of Eq. A.1 that satisfies Eqs. A.5.

To solve this problem, we introduce the transform

$$y(x,t) = y_1(x,t) + y_2(x,t), \tag{A.6}$$

such that

$$(\beta^2U^2 - \bar{v}^2)\frac{\partial^2 y_1}{\partial x^2} + 2\beta^2 U \frac{\partial^2 y_1}{\partial x \partial t} + \frac{\partial^2 y_1}{\partial t^2} = 0 \tag{A.7}$$

for

$$\left.\begin{array}{ll} y_1 = g(x) & \text{at } t = 0, \\ \text{and} & \\ \dfrac{\partial y_1}{\partial t} = h(x) & \text{at } t = 0; \end{array}\right\} \tag{A.8}$$

and

$$(\beta^2U^2 - \bar{v}^2)\frac{\partial^2 y_2}{\partial x^2} + 2\beta U \frac{\partial^2 y_2}{\partial x \partial t} + \frac{\partial^2 y_2}{\partial t^2} = R(x,t) \tag{A.9}$$

for

$$\left.\begin{array}{ll} y_2 = 0 & \text{at } t = 0, \\ \text{and} & \\ \dfrac{\partial y_2}{\partial t} = 0 & \text{at } t = 0. \end{array}\right\} \tag{A.10}$$

The solution of Eqs. A.7 and A.8 is of the form

$$y_1 = G(x - \bar{v}_1 t) + H(x + \bar{v}_2 t). \tag{A.11}$$

We find immediately that

$$G(x) + H(x) = g(x) \tag{A.12a}$$

and

$$-\bar{v}_1 \frac{dG}{dx} + \bar{v}_2 \frac{dH}{dx} = h(x). \tag{A.12b}$$

Integration of Eq. A.12b gives

$$-\bar{v}_1 G(x) + \bar{v}_2 H(x) = \int_0^x h(s)\, ds + C, \tag{A.13}$$

where C is a constant. From Eqs. A.12a and A.13, we find

$$\left.\begin{aligned} G(x) &= \frac{1}{\bar{v}_1 + \bar{v}_2}\left[v_2 g(x) - \int_0^x h(s)\, ds - C\right] \\ &\text{and} \\ H(x) &= \frac{1}{\bar{v}_1 + \bar{v}_2}\left[v_1 g(x) + \int_0^x h(x)\, dx + C\right]. \end{aligned}\right\} \tag{A.14}$$

Substitution of the values found for G and H in Eq. A.11 yields

$$y_1 = \frac{1}{v_1 + v_2}\left[\bar{v}_1 g(x+\bar{v}_2 t) + \bar{v}_2 g(x - \bar{v}_1 t) + \int_{x-\bar{v}_1 t}^{x+\bar{v}_2 t} h(s)\, ds\right], \tag{A.15}$$

which is D'Alembert's formula for the solution of the initial-value problem for the wave equation.

To solve Eq. A.9 with initial conditions A.10, we introduce the new independent variables ξ_1 and ξ_2, by means of the substitution

$$\left.\begin{aligned} \xi_1 &= x - \bar{v}_1 t \\ &\text{and} \\ \xi_2 &= x + \bar{v}_2 t. \end{aligned}\right\} \tag{A.16}$$

Equation A.9 then reduces to

$$\frac{\partial^2 y_2}{\partial \xi_1 \partial \xi_2} = -\bar{R}(\xi_1, \xi_2), \tag{A.17}$$

where

$$\bar{R}(\xi_1, \xi_2) = \frac{1}{(\bar{v}_1 + \bar{v}_2)^2} R\left[\frac{v_1 \xi_2 + v_2 \xi_1}{2}, \frac{\xi_2 - \xi_1}{2(\bar{v}_1 + \bar{v}_2)}\right]. \tag{A.18}$$

The transform takes the upper half of the (x, t)-plane, $t > 0$, into the part $\xi_2 \geq \xi_1$ of the (ξ_1, ξ_2)-plane. The initial conditions for $y_2(x, t)$ at $t = 0$, lead to the initial conditions

$$y_2 = 0, \quad \frac{\partial y_2}{\partial \xi_1} = \frac{\partial y_2}{\partial \xi_2} = 0,$$

on the line $\xi_1 = \xi_2$ in the (ξ_1, ξ_2)-plane. Our problem is now to determine y_2 from these initial values in the half-plane $\xi_2 - \xi_1 > 0$. This is done by integrating Eq. A.17 twice, first with respect to ξ_1 and then with respect to ξ_2, starting both times at the point on the line $\xi_1 = \xi_2$. The result is

$$y_2 = \int_{\xi_2'=\xi_2}^{\xi_1} \int_{\xi_1'=\xi_2'}^{\xi_1} \overline{R}(\xi_1', \xi_2')\, d\xi_1'\, d\xi_2'. \tag{A.19}$$

Equation A.19 can be easily transformed by reintroducing x and t as independent variables. From Eq. A.16, it is seen that

$$d\xi_1\, d\xi_2 = (\overline{v}_1 + \overline{v}_2)\, dx\, dt;$$

therefore, we obtain a solution of the problem in terms of the variables x and t:

$$y_2 = \frac{1}{\overline{v}_1 + \overline{v}_2} \int_0^t d\tau \int_{x-\overline{v}_1(t-\tau)}^{x+\overline{v}_2(t-\tau)} R(s,\tau)\, ds. \tag{A.20}$$

Finally, the sum of Eqs. A.15 and A.20 gives the solution of Eq. A.1:

$$y = \frac{1}{\overline{v}_1 + \overline{v}_2} \left[\overline{v}_1 g(x + \overline{v}_2 t) + \overline{v}_2 g(x - \overline{v}_1 t) + \int_{x-\overline{v}_1 t}^{x+\overline{v}_2 t} h(s)\, ds \right.$$

$$\left. + \int_0^t d\tau \int_{x-\overline{v}_1(t-\tau)}^{x+\overline{v}_2(t-\tau)} R(s,\tau)\, ds \right]. \tag{A.21}$$

When U = 0, Eq. A.21 reduces to classical solution of the wave equation.[10]

APPENDIX B

Equations Determining Stability-Instability Boundaries

This appendix lists the equations that determine the values of χ_1, χ_2, τ_1, and τ_2.

Taking two-mode approximation, the expansion of Eq. 69 gives

$$|\alpha_{ij}| = 0, \quad i,j = 1, 2, 3, 4, \tag{B.1}$$

where

$$\left.\begin{aligned}
\alpha_{11} &= 1 - (1 - \mu + \tfrac{1}{2}\mu^2)\, r_1^2 - \left(\frac{\omega}{2\nu_1}\right)^2, \\
\alpha_{12} &= 0, \\
\alpha_{13} &= -\zeta_1 \frac{\omega}{\nu_1}, \\
\alpha_{14} &= \tfrac{8}{3}\beta u \frac{\omega}{\nu_1^2}, \\
\alpha_{21} &= 0, \\
\alpha_{22} &= 1 - (1 - \mu + \tfrac{1}{2}\mu^2)\, r_2^2 - \left(\frac{\omega}{2\nu_2}\right)^2, \\
\alpha_{23} &= -\tfrac{8}{3}\beta u \frac{\omega}{\nu_2^2}, \\
\alpha_{24} &= -\zeta_2 \frac{\omega}{\nu_2}, \\
\alpha_{31} &= \zeta_1 \frac{\omega}{\nu_1}, \\
\alpha_{32} &= -\tfrac{8}{3}\beta u \frac{\omega}{\nu_1^2}, \\
\alpha_{33} &= 1 - (1 + \mu + \tfrac{1}{2}\mu^2)\, r_1^2 - \left(\frac{\omega}{2\nu_1}\right)^2, \\
\alpha_{34} &= 0, \\
\alpha_{41} &= \tfrac{8}{3}\beta u \frac{\omega}{\nu_2^2}, \\
\alpha_{42} &= \zeta_2 \frac{\omega}{\nu_2}, \\
\alpha_{43} &= 0, \\
&\text{and} \\
\alpha_{44} &= 1 - (1 + \mu + \tfrac{1}{2}\mu^2)\, r_2^2 - \left(\frac{\omega}{2\nu_2}\right)^2.
\end{aligned}\right\} \tag{B.2}$$

For a given system, the only unknown in Eq. B.1 is the frequency ω. From this equation, we can determine the values of $\left(\frac{\omega}{2\nu_1}\right)^2$ and denote them by χ_1 and χ_2. Thus the principal instability region is specified by

$$\sqrt{\chi_2} \le \frac{\omega}{2\nu_1} \le \sqrt{\chi_1}. \tag{B.3}$$

Similarly, the equation to determine boundaries of the second instability region can be shown as

$$|\beta_{ij}| = 0, \qquad i,j = 1, 2, 3, 4, 5, 6, \tag{B.4}$$

where

$$\left.\begin{aligned}
\beta_{11} &= 1 - (1+\tfrac{1}{4}\mu^2)\, r_1^2 - \left(\frac{\omega}{\nu_1}\right)^2, \\
\beta_{12} &= 0, \\
\beta_{13} &= 0, \\
\beta_{14} &= \tfrac{8}{3}\mu\beta u \frac{\omega}{\nu_1^2}, \\
\beta_{15} &= -2\zeta_1 \frac{\omega}{\nu_1}, \\
\beta_{16} &= \tfrac{16}{3}\beta u \frac{\omega}{\nu_1^2}, \\
\beta_{21} &= 0, \\
\beta_{22} &= 1 - (1+\tfrac{1}{4}\mu^2)\, r_2^2 - \left(\frac{\omega}{\nu_2}\right)^2, \\
\beta_{23} &= -\tfrac{8}{3}\mu\beta u \frac{\omega}{\nu_2^2}, \\
\beta_{24} &= 0, \\
\beta_{25} &= -\tfrac{16}{3}\beta u \frac{\omega}{\nu_2^2}, \\
\beta_{26} &= -2\zeta_2 \frac{\omega}{\nu_2}, \\
\beta_{31} &= 0, \\
\beta_{32} &= -\tfrac{4}{3}\mu\beta u \frac{\omega}{\nu_1^2}, \\
\beta_{33} &= 1 - (1+\tfrac{1}{2}\mu^2)\, r_1^2, \\
\beta_{34} &= 0, \\
\beta_{35} &= -\mu r_1^2, \\
\beta_{36} &= 0, \\
\beta_{41} &= \tfrac{4}{3}\mu\beta u \frac{\omega}{\nu_2^2}, \\
\beta_{42} &= 0, \\
\beta_{43} &= 0, \\
\beta_{44} &= 1 - (1+\tfrac{1}{2}\mu^2)\, r_2^2, \\
\beta_{45} &= 0, \\
\beta_{46} &= -\mu r_2^2, \\
\beta_{51} &= 2\zeta_1 \frac{\omega}{\nu_1}, \\
\beta_{52} &= -\tfrac{16}{3}\beta u \frac{\omega}{\nu_1^2}, \\
\beta_{53} &= -2\mu r_1^2, \\
\beta_{54} &= 0, \\
\beta_{55} &= 1 - (1+\tfrac{3}{4}\mu^2)\, r_1^2 - \left(\frac{\omega}{\nu_1}\right)^2, \\
\beta_{56} &= 0, \\
\beta_{61} &= \tfrac{16}{3}\beta u \frac{\omega}{\nu_2^2}, \\
\beta_{62} &= 2\zeta_2 \frac{\omega}{\nu_2}, \\
\beta_{63} &= 0, \\
\beta_{64} &= -2\mu r_2^2, \\
\beta_{65} &= 0, \text{ and} \\
\beta_{66} &= 1 - (1+\tfrac{3}{4}\mu^2)\, r_2^2 - \left(\frac{\omega}{\nu_2}\right)^2.
\end{aligned}\right\} \tag{B.5}$$

We assign τ_1 and τ_2 for $\left(\frac{\omega}{\nu_1}\right)^2$, which satisfy Eq. B.5. The second instability region is given by

$$\sqrt{\tau_2} \leq \frac{\omega}{\nu_1} \leq \sqrt{\tau_1}. \tag{B.6}$$

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